

Constraining the Noncommutative Spectral Action via Astrophysical Observations

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The noncommutative spectral action extends our familiar notion of commutative spaces, using the data encoded in a spectral triple on an almost commutative space. Varying a rather simple action, one can derive all of the standard model of particle physics in this setting, in addition to a modified version of Einstein-Hilbert gravity. In this letter we use observations of pulsar timings, assuming that no deviation from General Relativity has been observed, to constrain the gravitational sector of this theory. Whilst the bounds on the coupling constants remain rather weak, they are comparable to existing bounds on deviations from General Relativity in other settings and are likely to be further constrained by future observations.

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INTRODUCTION

Approaching Planckian energies, the assumption of Riemannian geometry and the validity of General Relativity (GR) break down and one is forced to describe the space-time geometry within a fully quantum framework. NonCommutative Geometry (NCG) [1, 2] is based on the idea that as we approach Planckian energy scales, our intuitive description of space-time being a commutative manifold ceases to be a valid approximation. In its simplest but nevertheless powerful version, NCG implies that slightly below Planck energy, space-time is well approximated by the product of a four-dimensional smooth compact Riemannian manifold \mathcal{M} and a finite noncommutative space \mathcal{F} . Such spaces are called “almost commutative” spaces and they are the simplest extensions of the commutative spaces we use in GR. This is a strong assumption which is certainly expected to break at the Planck scale, where the notion of classical geometry loses all meaning, however at low energies it should be a good approximation.

The noncommutative nature of \mathcal{F} is given by the real spectral triple $(\mathcal{A}, \mathcal{H}, D)$, where \mathcal{A} is an involution of operators on the Hilbert space \mathcal{H} , and D is a self-adjoint unbounded operator in \mathcal{H} . The algebra \mathcal{A} is the algebra of coordinates, the operator D corresponds to the inverse line element of Riemannian geometry, and the commutator $[D, a]$ with $a \in \mathcal{A}$ plays the rôle of the differential quotient da/ds , with ds the unit of length. The resulting physical Lagrangian is obtained from the asymptotic expansion in the energy scale Λ of a spectral action functional of the form $\text{Tr}(f(D/\Lambda))$ defined on noncommuta-

tive spaces, where f is a cut-off function (i.e. a test function of compact support). The coupling with fermions can be obtained by including an additional term in the spectral action functional. The choice of the finite dimensional algebra is the underlying geometric input which determines the physical implications of the model, in particular the particle content of the theory.

The NCG spectral action offers a simple and elegant explanation for the phenomenology of the Standard Model (SM) compatible with right-handed neutrinos and neutrino masses [3] (the approach has also been used to derive supersymmetric extensions to the standard model [4]). This approach to the SM has been proposed as a way to achieve unification, based on the symplectic-unitary group (the algebra constructed in $\mathcal{M} \times \mathcal{F}$ is assumed to be symplectic-unitary) in the Hilbert space, instead of the finite dimensional Lie groups. Note that the NCG spectral action is a classical theory which, in principle, should eventually be quantized. Whilst an understanding of how to quantize such noncommutative spaces has not yet been fully developed, already at the classical level the theory introduces several extensions to standard GR. Specifically, the physical Lagrangian contains, in addition to the full SM Lagrangian, the Einstein-Hilbert action with a cosmological term, a topological term related to the Euler characteristic of the space-time manifold, a conformal Weyl term and a conformal coupling of the Higgs field to gravity. In contrast to the SM on a fixed background, the coefficients of the gravitational terms in this NCG action depend on the Yukawa parameters of the particle physics content.

The parameters of the NCG spectral action model are set at the scale Λ , considered to be the unification scale, while physical predictions at lower energies are recovered by running the parameters down through Renormalization Group Equations (RGE). Thus, the spectral action at the unification scale Λ is directly applicable to early universe cosmological models [5–8], while extrapolations

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to lower energies can be obtained via RGE and inclusion of nonperturbative effects in the spectral action.

The gravitational part of the asymptotic formula for the bosonic sector of the NCG spectral action, including the coupling between the Higgs field and the Ricci curvature scalar, reads [3]

$$\mathcal{S}_{\text{grav}} = \int \left(\frac{1}{2\kappa_0^2} R + \alpha_0 C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + \tau_0 R^* R^* - \xi_0 R |\mathbf{H}|^2 \right) \sqrt{-g} d^4x ; \quad (1)$$

\mathbf{H} is a rescaling $\mathbf{H} = (\sqrt{af_0}/\pi)\phi$ of the Higgs field ϕ to normalize the kinetic energy, the momentum f_0 is physically related to the coupling constants at unification and the coefficient a is related to the fermion and lepton masses and lepton mixing. Note that we are using conventions in which the signature is $(-, +, +, +)$ and the Ricci tensor is defined as $R_{\mu\nu} = R^\rho{}_{\mu\nu\rho}$, with $R_{\mu\nu\rho}{}^\sigma \omega_\sigma = [\nabla_\mu, \nabla_\nu] \omega_\rho$. In the above action, Eq. (1), the first term is the familiar Einstein-Hilbert term, the second one is a Weyl curvature term, the third term $R^* R^* = (1/4) \epsilon^{\mu\nu\rho\sigma} \epsilon_{\alpha\beta\gamma\delta} R_{\mu\nu}^{\alpha\beta} R_{\rho\sigma}^{\gamma\delta}$, is the topological term that integrates to the Euler characteristic, hence nondynamical, and the last one couples gravity to the Higgs field¹ and can have important consequences at high energies, such as in the early universe [5–8]. Here, we will be concerned with the low energy, weak curvature regime where this term is small.

Neglecting the nonminimal coupling between the Higgs field and the Ricci curvature, the equations of motion derived from the spectral action above read [5]

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R + \frac{1}{\beta^2} \left[2C_{\lambda;\kappa}^{\mu\lambda\nu\kappa} + C^{\mu\lambda\nu\kappa} R_{\lambda\kappa} \right] = 8\pi G T_{\text{matter}}^{\mu\nu} , \quad (2)$$

where β^2 is defined as $\beta^2 = -1/(32\pi G\alpha_0)$. Notice in particular that the NCG corrections vanish for Friedmann-Lemaître-Robertson-Walker (FLRW) cosmologies [5] and Schwarzschild solutions, which makes it difficult to place restrictions on these terms via cosmology or solar-system tests. The best constraint on, different *ad hoc*, curvature squared terms are obtained from measurements of the orbital precession of Mercury, imposing a rather weak lower bound on β , namely $\beta > 3.2 \times 10^{-9} \text{m}^{-1}$ [9]. This constraint was however found for terms of different form (but of the same order) to the Weyl term appearing in the NCG spectral action approach we investigate here. In what follows, we will specifically study how one can constrain β within the NCG context. The parameter β can be equivalently expressed in terms of f_0 , through

$\beta^2 = (5\pi)/(48Gf_0)$, so by imposing a lower limit to β , we actually set an upper limit to the moment f_0 of the cut-off function used to define the spectral action. The normalization of kinetic terms in the spectral action imposes the following relation between the gauge couplings of the Standard Model, g_1, g_2, g_3 and the coefficient f_0 [3], namely $g_3^2 f_0 / (2\pi^2) = 1/4$, $g_3^2 = g_2^2 = (5/3)g_1^2$. The importance of constraining β is thus clear, since f_0 can be used to specify the initial conditions on the gauge couplings, a constraint on β corresponds to a restriction on the particle physics at unification.

We will study the energy lost to gravitational radiation by orbiting binaries, so we consider the weak field limit of Eq. (2). The general first order solution for a perturbation against a Minkowski background is [10]

$$h^{\mu\nu}(\mathbf{r}, t) = \frac{4G\beta}{c^4} \int d\mathbf{r}' dt' \frac{\Theta(T)}{\sqrt{(cT)^2 - |\mathbf{R}|^2}} \times \mathcal{J}_1 \left(\beta \sqrt{(cT)^2 - |\mathbf{R}|^2} \right) T^{\mu\nu}(\mathbf{r}', t') \Theta(cT - |\mathbf{R}|) ; \quad (3)$$

$T = t - t'$ is the difference between the time of observation (t) and emission (t') of the perturbation, $\mathbf{R} = \mathbf{r} - \mathbf{r}'$ is the difference between the locations of the observer (\mathbf{r}) and emitter (\mathbf{r}'), \mathcal{J}_1 is a Bessel function of the first kind and Θ is the Heavyside step function. In the far field limit, $|\mathbf{r}| \approx |\mathbf{r} - \mathbf{r}'|$, the spatial components of Eq. (3) become

$$h^{ik}(\mathbf{r}, t) \approx \frac{2G\beta}{3c^4} \int_{-\infty}^{t - \frac{1}{c}|\mathbf{r}|} \frac{dt'}{\sqrt{c^2(t-t')^2 - |\mathbf{r}|^2}} \times \mathcal{J}_1 \left(\beta \sqrt{c^2(t-t')^2 - |\mathbf{r}|^2} \right) \ddot{D}^{ik}(t') , \quad (4)$$

where we have, introduced the quadrupole moment,

$$D^{ik}(t) \equiv \frac{3}{c^2} \int d\mathbf{r} x^i x^k T^{00}(\mathbf{r}, t) . \quad (5)$$

From Eq. (2) is it clear that this theory reduces to that of GR in the $\beta \rightarrow \infty$ limit, and one can check that in this limit Eq. (4) does indeed reproduce the standard result for a massless graviton. For finite β however, one finds that gravitational radiation contains both massive and massless modes, both of which are sourced from the quadrupole moment of the system.

GRAVITATIONAL RADIATION FROM CIRCULAR BINARIES

We will derive the explicit formula for the energy lost to gravitational radiation from a binary pair in a circular orbit. One can similarly consider binaries in elliptical orbits; for simplicity we consider only circular Keplerian orbits. Similarly, we neglect effects due to the internal

¹ Such a term should always be present when one considers gravity coupled to scalar fields.

structure of the bodies as well as local astrophysical effects, such as mass transfer, tidal stripping *etc.*, focusing instead on the purely gravitational consequences of NCG.

Consider a circular binary pair, of masses m_1, m_2 . For such a system, orbiting in the xy -plane, the only nonzero components of the quadrupole moment are [11]

$$\begin{aligned}\ddot{D}^{xx}(t) &= 12\mu|\rho|^2 \sin(2\psi(t))\omega^3 \\ &= -\ddot{D}^{yy}(t), \\ \ddot{D}^{xy}(t) &= -12\mu|\rho|^2 \cos(2\psi(t))\omega^3, \\ D^{zz} &= -\mu|\rho|^2,\end{aligned}\quad (6)$$

where $\mu = m_1 m_2 / (m_1 + m_2)$ is the reduced mass of the system, $|\rho|$ is the magnitude of the separation vector between the bodies, which is constant for circular orbits, ψ is the angle of the bodies relative to the x -axis and $\omega = \dot{\psi}$ is the orbital frequency, which for this simple system is a constant given by

$$\omega \equiv \dot{\psi} = |\rho|^{-3/2} \sqrt{G(m_1 + m_2)}. \quad (7)$$

Following the standard approach (see e.g., Ref. [11]), the rate of energy loss, in the far field limit, is

$$-\frac{d\mathcal{E}}{dt} \approx \frac{c^2}{20G} |\mathbf{r}|^2 \dot{h}_{ij} \dot{h}^{ij}. \quad (8)$$

This allows us to explicitly test the theory by comparing this prediction to binary pulsar measurements, for which the energy loss has been very well characterized (see, Table I) and hence constrain β .

Using Eq. (4) one finds [10]

$$\begin{aligned}\dot{h}^{ij} \dot{h}_{ij} &= \frac{128\mu^2 |\rho|^4 \omega^6 G^2 \beta^2}{c^8} \\ &\times \left[f_c^2 \left(\beta|\mathbf{r}|, \frac{2\omega}{\beta c} \right) + f_s^2 \left(\beta|\mathbf{r}|, \frac{2\omega}{\beta c} \right) \right],\end{aligned}\quad (9)$$

where we have defined the functions:

$$\begin{aligned}f_s(x, z) &\equiv \int_0^\infty \frac{ds}{\sqrt{s^2 + x^2}} \mathcal{J}_1(s) \sin(z\sqrt{s^2 + x^2}), \\ f_c(x, z) &\equiv \int_0^\infty \frac{ds}{\sqrt{s^2 + x^2}} \mathcal{J}_1(s) \cos(z\sqrt{s^2 + x^2})\end{aligned}\quad (10)$$

The integrals above, Eq. (10), exhibit a strong resonance behavior at $z = 1$, however they are easily evaluated for both $z < 1$ and $z > 1$. This resonance corresponds to a critical frequency given by

$$2\omega_c = \beta c, \quad (11)$$

and we can expect strong deviations from the standard results of GR for orbital frequencies close to this critical frequency.

One can evaluate numerically the functions in Eq. (10) and fit them to an explicit functional form. Thus, for

$\omega < \omega_c$ one obtains

$$\begin{aligned}&\left[f_c \left(\beta|\mathbf{r}|, \frac{\omega}{\omega_c} \right) \right]^2 + \left[f_s \left(\beta|\mathbf{r}|, \frac{\omega}{\omega_c} \right) \right]^2 \\ &\approx \frac{1}{(\beta|\mathbf{r}|)^2} \exp \left(\frac{C}{\beta|\mathbf{r}| \left(1 - \frac{\omega}{\omega_c} \right)} \mathcal{J}_1 \left(\beta|\mathbf{r}| - \frac{\omega}{\omega_c} \right) \right),\end{aligned}\quad (12)$$

where C is approximately a constant, $C \approx 0.175$, except as ω approaches ω_c . Similarly, for $\omega > \omega_c$ one gets

$$\left[f_c \left(\beta|\mathbf{r}|, \frac{\omega}{\omega_c} \right) \right]^2 + \left[f_s \left(\beta|\mathbf{r}|, \frac{\omega}{\omega_c} \right) \right]^2 \quad (13)$$

$$\approx \frac{4}{(\beta|\mathbf{r}|)^2} \sin^2 \left(\beta|\mathbf{r}| \left(\tilde{f} \left(\frac{\omega}{\omega_c} \right) \right)^{-1} \right), \quad (14)$$

where the function \tilde{f} is approximately

$$\tilde{f} \left(\frac{\omega}{\omega_c} \right) \approx 4 \sqrt{\left(\frac{\omega}{\omega_c} \right)^2 - 1} + 2 \exp \left(-\sqrt{\left(\frac{\omega}{\omega_c} \right)^2 - 1} \right). \quad (15)$$

Note that as we will see, the precise form of this function is unimportant. See Ref. [10] for a discussion on the accuracy of the approximations.

Using the above approximations, one can expand Eq. (8) in the large distance (large $|\mathbf{r}|$) limit, to find the rate of energy lost to gravitational radiation:

$$\begin{aligned}-\frac{d\mathcal{E}}{dt} &\approx \frac{32G\mu^2 \rho^4 \omega^6}{5c^5} \\ &\times \begin{cases} 1 + \frac{C}{\beta|\mathbf{r}| \left(1 - \frac{\omega}{\omega_c} \right)} \mathcal{J}_1 \left(\beta|\mathbf{r}| - \frac{\omega}{\omega_c} \right) + \dots & ; \omega < \omega_c \\ 4 \sin^2 \left(\beta|\mathbf{r}| \tilde{f} \left(\frac{\omega}{\omega_c} \right) \right) & ; \omega > \omega_c \end{cases},\end{aligned}\quad (16)$$

where in the $\omega < \omega_c$ case the dots refer to higher powers of $1/(\beta|\mathbf{r}|)$. Thus, for orbital frequencies small compared to ω_c , any deviation from the standard result is suppressed by the distance to the source. Notice that in this case, the $\beta \rightarrow \infty$ (i.e., $\alpha_0 \rightarrow 0$) limit reproduces the GR result, as it should. For the $\omega > \omega_c$ case, the result would only agree with the General Relativistic result if $\beta|\mathbf{r}| \tilde{f}(\omega/\omega_c) = \pi/3$, which is clearly not true for systems at different distances, $|\mathbf{r}|$, with different orbital frequencies, ω . Hence, we can immediately eliminate the $\omega > \omega_c$ case, simply by noting that observations of the energy lost to gravitational radiation agree, to a high level of accuracy, with those of GR for many different systems.

The resonance appearing in Eq. (10) leads to a simple heuristic argument to rule out the $\omega > \omega_c$ case. A system with $\omega < \omega_c$ cannot increase its orbital frequency above ω_c , without losing a significant fraction of its energy to gravitational radiation. Similarly, a system with $\omega > \omega_c$ cannot decrease its orbital frequency across this

boundary. Since one expects all astrophysical systems to have formed from the coalescence of relatively cold, slowly moving systems, it is reasonable to suppose that at some time in the past, all binary systems had very slowly varying quadrupole moments and hence that $\omega < \omega_c$. In the following, we will see that this restriction places a strong constraint on β (which defines ω_c).

For the physically interesting case of $\omega < \omega_c$, the amplitude of the deviation from the standard result is small, due to the $1/|\mathbf{r}|$ suppression, however there are two interesting features: firstly, the existence of a critical frequency ω_c and secondly, the fact that the rate of flux of gravitational radiation is oscillatory.

The critical frequency comes from the fact that this theory contains a natural frequency scale given by $\beta c \sim c(-\alpha_0 G)^{-1}$. This scale is set by the moments of the cut-off function used to define the spectral action. Physically one can think of this as the scale at which noncommutative effects become dominant. What is important for this work, is that the binary systems must have orbital frequencies below this critical value, since otherwise the theory would predict significant deviations from the results of standard GR, which can be ruled out observationally.

The presence of the Bessel function in Eq. (16) means that the amplitude of the deviation from the standard result of GR will oscillate both with changing distances and changing frequencies. Whilst such correlations may present new observational signatures, the effect is heavily suppressed by the $|\mathbf{r}|^{-1}$ factor in Eq. (16); here we focus on overall amplitude of deviations from the GR result.

ASTROPHYSICAL CONSTRAINTS

Having calculated the general form of the gravitational radiation from binary systems, within this NCG theory of gravity, we can now constrain the main parameter of the theory, β , via observational data. From Eq. (16) it is clear that the only data needed is the orbital frequency and the distance to the binary system. If one had considered elliptical binary orbits, additional parameters would come into play, however we are concerned only with the order of magnitude of the constraint and hence neglect such additional complications. A more detailed quantitative analysis would require the inclusion of the ellipticity as well as various other near field effects.

Table I gives the binaries we are considering. We focus on binary pulsars for which the rate of change of the orbital frequency has been well characterized. In all these cases the predictions of GR agree with the data to high accuracy (see, Table I). We can thus restrict β by requiring that the magnitude of deviations from GR, given by Eq. (16), be less than this uncertainty.

Using the data on the six binaries given in Table I, and requiring that $\omega < \omega_c$ (see, the discussion above), we find $\beta > 7.55 \times 10^{-13} \text{ m}^{-1}$. The restrictions coming from the

Binary	Distance (pc)	Orbital Period (hr)	Eccentricity	GR (%)
PSR J0737-3039	~ 500	2.454	0.088	0.2
PSR J1012-5307	~ 840	14.5	$< 10^{-6}$	10
PSR J1141-6545	> 3700	4.74	0.17	6
PSR B1916+16	~ 6400	7.752	0.617	0.1
PSR B1534+12	~ 1100	10.1	?	1
PSR B2127+11C	~ 9980	8.045	0.68	3

TABLE I: We calculate the constraint on the NCG theory of gravity, via the predicted energy lost to gravitational radiation from the above binaries (Refs. [12], [13], [14], [15], [16], [17], respectively). The column marked GR, indicates the approximate accuracy to which the rate of change of the orbital period agrees with the predictions of GR.

PSR J0737-3039	$\beta > 7.55 \times 10^{-13} \text{ m}^{-1}$
PSR J1012-5307	$\beta > 7.94 \times 10^{-14} \text{ m}^{-1}$
PSR J1141-6545	$\beta > 3.90 \times 10^{-13} \text{ m}^{-1}$
PSR B1913+16	$\beta > 2.39 \times 10^{-13} \text{ m}^{-1}$
PSR B1534+12	$\beta > 1.83 \times 10^{-13} \text{ m}^{-1}$
PSR B2127+11C	$\beta > 2.30 \times 10^{-13} \text{ m}^{-1}$

TABLE II: For each binary system, we restrict β by requiring that the energy lost to gravitational radiation agrees with the prediction of GR to within observational uncertainties.

individual systems are given in Table II.

Due to the large distances to these systems, the constraint is almost exactly due to $\omega < \omega_c$ which, using the definition of ω_c given in Eq. (11), becomes $\beta > 2\omega/c$. Thus, the strongest constraint comes from systems with high orbital frequencies. This will be true for all systems for which $2\omega|\mathbf{r}|/c$ is large. Future observations of rapidly orbiting binaries, relatively close to the Earth, could thus improve this constraint by many orders of magnitude.

This dependence of the constraint on the orbital frequency, suggests that other astrophysical objects, with high frequency periodicity, such as individual pulsars or merger in-spirals may provide a more stringent constraint. Whilst the analysis given here is only applicable to binaries, it can be extended by replacing Eq. (6) by the quadrupole moments of whatever system is of interest.

CONCLUSIONS

General Relativity is formulated within the arena of Riemannian geometry a natural extension of which is NonCommutative Geometry. The spectral action approach produces all the Standard Model fields as well as gravitational terms, from purely geometric considerations. Thus, both gravity and matter are treated in a

similar manner within NCG, which also provides us with concrete relationships between matter and gravitational couplings. The asymptotic expansion of the gravitational sector of this theory produced modifications to GR and in this paper we use these modifications to test and constrain the theory through observations.

We have considered the energy lost by circular binary systems to gravitational radiation and shown that for the predicted values to agree with observations, a key parameter of the theory can be constrained. We have focused on binary pulsar systems, for which the rate of change of the orbital frequency is well known and explicitly calculated the predicted deviation from the GR expressions. We have shown that this restricts the value of the Weyl squared coupling in the bosonic action (*i.e.*, α_0 in Eq. (1)). This observational constraint may seem rather weak, requiring only that $\beta \geq 7.55 \times 10^{-13} \text{m}^{-1}$, however it is comparable to (but larger than) existing constraints on similar, *ad hoc*, additions to GR. In particular, constraints on additions to the Einstein-Hilbert action, of the form R^2 and $R_{\mu\nu}R^{\mu\nu}$, are of the order of $\beta_{R^2} \geq 3.2 \times 10^{-9} \text{m}^{-1}$, where β_{R^2} is the β parameter associated with the couplings of these terms [9]. Whilst our constraint is several orders of magnitude weaker than these, it will rapidly be improved as more binary pulsars are discovered and the observations of existing systems improve. This is to be contrasted with the existing constraints which rely on the perihelion precession of Mercury, the accuracy of which is unlikely to improve significantly in the future.

As an example, white dwarf binaries reach orbital frequencies of the order of $\sim (10 - 100) \text{mHz}$ towards the end of their merger, whilst neutron binaries reach $\sim (10 - 100) \text{Hz}$ and should be readily observable with the *Laser Interferometer Space Antenna* (LISA) [18]. Whilst such systems would require a greater understanding of the strong and near field effects than that presented here, one can expect the constraint on β coming from such objects to be of the order of $\beta > (10^{-10} - 10^{-6}) \text{m}^{-1}$.

We were able to constrain the natural length, defined through the $f_0 = f(0)$ momentum of the cut-off function f — a real parameter related to the coupling constants at unification — at which the noncommutative effects become dominant, by purely astrophysical observations.

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